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Blow-up in the Boussinesq equation

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The Boussinesq equation is known to be the fundamental nonlinear model, describing wave propagation in a weakly dispersive nonlinear medium. In this paper, we prove the existence of the collapse dynamics for the two basic forms of the Boussinesq equation in the case of the *periodic* boundary conditions. The sufficient criterion of the blow-up is found analytically.

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I. INTRODUCTION

It is known that the propagation of nonlinear pulses in a weakly dispersive medium is governed by the Boussinesq equation [1, 2], which can be written in the following two basic forms:

$$U_{tt} - U_{xx} - U_{xxtt} + (U^2)_{xx} = 0 \quad (1)$$

and

$$U_{tt} - U_{xx} + U_{xxxx} + (U^2)_{xx} = 0. \quad (2)$$

The linear parts of Eqs. (1) and (2) determine dispersion relations of the linear waves in two different cases. Equation (1) corresponds to the negative wave dispersion: $\omega^2 = k^2/(1+k^2)$. Respectively, Eq. (2) is derived for linear waves with the positive dispersion: $\omega^2 = k^2 + k^4$. By a sign of the dispersion one means that in the long-wavelength limit ($k \to 0$) the dispersion relation of the linear waves is close to that for the sound waves, i.e., $\omega = k(1 \pm k^2)$, and by definition, upper and lower signs correspond to the positive and negative dispersions, respectively.

Equations (1) and (2) are the universal model for nonlinear wave dynamics in weakly dispersive media. Equation (1) is the so-called "improved" Boussinesq equation [3–6]. The term "improved" means that, in comparison with the "ill-posed" Boussinesq equation

$$U_{tt} - U_{xx} - U_{xxxx} + (U^2)_{xx} = 0, \quad (3)$$

for which the linear dispersion relation $\omega^2 = k^2 - k^4$ leads to a nonphysical instability of linear waves, $\mathrm{Im}(\omega) > 0$, for $k > 1$, Eq. (1) does not admit such a type of instability of linear modes. The effect of replacing ∂_x^2 by ∂_t^2 in the third term of Eq. (3) has been discussed in Ref. [6].

Equations (1) and (2) occur in a wide variety of physical systems [3-12]. They are of fundamental physical interest, because they describe the lowest-order (in terms of wave amplitudes) nonlinear effects in the evolution of perturbations with the dispersion relation close to that for the sound waves. For example, Eq. (1) describes a continuum limit of a one-dimensional nonlinear lattice [7], shallow-water waves [1,2] (see also [8]), a propagation of nonlinear acoustic waves on a circular rod [4, 9], solitons in the nonlinear electric transmission lines (see, e.g., Ref. [10]), and in other models supporting linear waves with the negative dispersion. Equation (2) is the so-called well-posed Boussinesq equation which occurs in the problems dealing with propagation of nonlinear waves in a medium with positive dispersion, for example, electromagnetic waves interacting with transversal optical phonons in nonlinear dielectrics [11], magnetosound waves in plasmas [12], and magnetoelastic waves in antiferromagnets [13]. It is known that for the transonic speed perturbations, by neglecting the interaction of waves moving in the opposite directions, Eqs. (1) and (2) may be reduced to the Korteveg–de Vries equation.

Equations (1) and (2) have been the subject of extensive investigations. The main features of these equations are rather well established. The well-posed Boussinesq equation (2) is integrable by the inverse scattering transform (IST) [14]. Dynamics of nonlinear waves in Eqs. (1) and (2) was studied in a number of papers, both numerically and analytically [3–12]. Stability of the bound states in the form of solitary waves and cnoidal waves were investigated in [15,16]. Numerical studies of the nonlinear dynamics for unstable solitons of Eq. (1) have demonstrated the possibility of the blow-up [9]. Blow-up [this term is generally used to refer to nonexistence of global solutions for the initial-value problem (see e.g., [20-22])] was proved for Eq. (2) in the case of special initial conditions [17]. It should be noted that the sufficient conditions for the blow-up in Eq. (2), which were obtained in Ref. [17], are not satisfied by soliton solutions of Eq. (2). An interesting fundamental issue is the coexistence of integrability of Eq. (2) by the IST with a possibility of the blow-up-type dynamics in this equation. As was found [15] by using the IST, certain soliton solutions of Eq. (2) may collapse for a finite time.

A time evolution of an arbitrary initial wave packet is one of the most important problems related to Eqs. (1) and (2). A question of the central interest to the analysis of an initial-value problem is whether or not the blow-up appears for given initial data. In Ref. [17] (see also [18]), by using the method suggested in Ref. [19], it was shown that for a large set of initial data there are no smooth solutions of the initial-value problem of Eq. (2) for all time. This approach is limited by the additional restriction on the initial distribution of perturbations, namely, the condition $\int U dx = 0$ was assumed to be fulfilled.

In most of the numerical investigations of the nonlinear dynamics given by Eqs. (1) and (2), the periodic boundary conditions were used. However, one may notice that there is no analytic proof of the blow-up for Eqs. (1) and (2) in the case of periodic boundary conditions. Our motivation in this work is to develop the method of analytic proof of blow-up to the case of *periodic* boundary conditions and to apply it to Eqs. (1) and (2). Thus, the primary objective of the present paper is to prove the blow-up in Eqs. (1) and (2) for the case of periodic boundary conditions and to obtain exact sufficient criteria of the collapse dynamics. In the limit when the spatial period L tends to infinity, the conditions of the blow-up for the infinite system are naturally recovered.

II. BASIC EQUATIONS AND CONSERVED QUANTITIES

In this section we discuss briefly general properties of Eqs. (1) and (2) and the method of majoring equations that will be used to prove the blow-up.

Equations (1) and (2) may be written as a first-order system in the Hamiltonian form. For Eq. (1) the Hamiltonian structure is given by

$$U_t = -\frac{\delta H_1}{\delta \Psi} = Q^{-1}\Psi_{xx}, \tag{4}$$

$$\Psi_t = \frac{\delta H_1}{\delta U} = U - U^2, \tag{5}$$

and, respectively, a similar representation takes place for Eq. (2):

$$U_t = -\frac{\delta H_2}{\delta \Phi} = \Phi_{xx}, \tag{6}$$

$$\Phi_t = \frac{\delta H_2}{\delta U} = U - U_{xx} - U^2. \tag{7}$$

Here non-negative operator Q is defined as

$$Q = -\frac{\partial^2}{\partial x^2} + 1.$$

Q^{-1} is the inverse operator to the Q. The corresponding Hamiltonians $H_1 = J_1 + J_2 - J_3$ and $H_2 = I_1 + I_2 + I_3 - I_4$ are of the form

$$H_1 = \tfrac{1}{2}\int_{-L}^{L} \Psi_x Q^{-1}\Psi_x dx + \tfrac{1}{2}\int_{-L}^{L} U^2 dx - \tfrac{1}{3}\int_{-L}^{L} U^3 dx \tag{8}$$

and

$$H_2 = \tfrac{1}{2}\int_{-L}^{L} \Phi_x^2 dx + \tfrac{1}{2}\int_{-L}^{L} U^2 dx + \tfrac{1}{2}\int_{-L}^{L} U_x^2 U dx - \tfrac{1}{3}\int_{-L}^{L} U^3 dx, \tag{9}$$

$2L$ being the system length.

Obviously, the Hamiltonians H_1 and H_2 are the integrals of motion of the corresponding dynamical systems.

Additionally, Eqs. (1) and (2) possess the following integrals of motion, which are generated by the invariance of the equations under space translation. For Eq. (1) the total momentum of the system takes the form

$$P_1 = \int_{-L}^{L} U\Psi_x dx, \tag{10}$$

and for Eq. (2), momentum is of the form

$$P_2 = \int_{-L}^{L} U\Phi_x dx. \tag{11}$$

It is easy to see also that the integral

$$M = \int_{-L}^{L} U dx \tag{12}$$

is a conserved quantity. This integral has the sense of the total "mass" of the perturbations or the total displacement and in the mathematical literature it is often referred to as the Casimir invariant. As we demonstrate below, the quantity M plays an important role in the nonlinear dynamics governed by Eqs. (1) and (2).

The main question we would like to answer in this paper is as follows: Which initial conditions to the Boussinesq equation lead to the collapse dynamics? To answer this question we use the so-called majoring equation method (see, e.g., Refs. [22,23]). This method works as follows. For a given partial differential equation, one considers an appropriate integral characteristic of solutions as a function of time. With a successful choice of such a quantity, one may obtain an ordinary differential equation or differential inequality to this function. Solving this equation (or differential inequality) one may find the conditions under which the function in question becomes infinite in a finite time. The most often used majoring inequality was suggested by Levine [19] and developed by Kalantarov and Ladyzhenskaya [17]. This majoring inequality is of the form

$$\Psi_{tt}\Psi - (1+\alpha)\Psi_t\Psi_t \geq 0, \tag{13}$$

where arbitrary coefficient α is assumed to be positive.

If at the initial moment $t = 0$ $\Psi(0) > 0$ and $\Psi_t(0) > 0$, then the function $\Psi \to \infty$ as
$t \to t_1 \leq \Psi(0)/\alpha\Psi_t(0)$.

To prove the blow-up in a distributed nonlinear system, it is enough to demonstrate that some appropriate function of time defined with the help of solutions of corresponding partial differential equation satisfies the inequality (13).

Although the method described above can be very useful, this procedure cannot be considered as a universal method, because there are no certain rules for the selection of the integral quantity which may be shown to become singular in a finite time. Nevertheless, to date this method is the most successfully used approach for an analytical proof of the blow-up. In this paper we basically follow the method developed in [17] and extend it to the case of the periodic boundary conditions. Throughout this paper we consider Eqs. (1) and (2) on the interval $x \in [-L, L]$ under the periodic boundary conditions.

III. WELL-POSED BOUSSINESQ EQUATION

First we consider the blow-up in the well-posed Boussinesq equation (2). For the initial-value problem on the interval $x \in (-\infty, +\infty)$, it was shown in [17] that the quantity $Z(t) = \int_{-\infty}^{+\infty} w^2 dx$ (where $w_x = U$) becomes infinite in a finite time under some additional assumptions.

This quantity, however, cannot be used as a majoring function in the case of the periodic boundary conditions, due to the lack of the periodicity of integrand function w. It will therefore be necessary to find a new candidate for this role.

Define the function f by the relation $f_x = U - \langle U \rangle$, where $\langle U \rangle$ stands for the mean value, obtained by period averaging, $\langle U \rangle = \frac{1}{2L}\int_{-L}^{L} U dx$. It is easy to check that the function f is periodic with the period $2L$. From Eq. (6) it follows that

$$f_t = \Phi_x. \tag{14}$$

Here we have used the fact that $\langle U \rangle$ is an integral of the motion of Eq. (2).

Consider now the time evolution of the following positive quantity $R(t) = \int_{-L}^{L} f^2 dx$:

$$\frac{\partial R}{\partial t} = 2\int_{-L}^{L} f\Phi_x dx. \tag{15}$$

By using the Cauchy-Schwarz inequality, one may obtain the following estimate which will be used later:

$$R_t = 2\int_{-L}^{L} f\Phi_x dx \leq 2R^{1/2}(2I_1)^{1/2}. \tag{16}$$

Differentiating Eq. (13) with respect to t, integrating by parts, and using Eq. (9), we come to the relation

$$\begin{aligned}\frac{\partial^2 R}{\partial t^2} &= 2\int_{-L}^{L} \Phi_x^2 dx - 2\int_{-L}^{L} (U - \langle U \rangle)(U - U_{xx} - U^2)dx \\ &= 4I_1 - 4I_2 - 4I_3 + 6I_4 \\ &\quad + 2\langle U \rangle \int_{-L}^{L} (U - U^2)dx.\end{aligned}$$

We are in a position now to demonstrate that the function $R(t)$ may be majorized from below by some function, which becomes singular in a finite time. Making use of the formulas (9) and (14), one may observe that R satisfies the following master inequality:

$$\begin{aligned}R_{tt}R - (1+\alpha)R_t^2 \geq R[&-6H_2 + 10I_1 + 2I_2 + 2I_3 \\ &-8(1+\alpha)I_1 + 4\langle U \rangle^2 L - 4\langle U \rangle I_2],\end{aligned}$$

so that, letting the free positive parameter $\alpha < \frac{1}{4}$, we find that

$$R_{tt}R - (1+\alpha)R_t^2 \geq R[-6H_2 + M^2L^{-1} + 2I_2(1 - ML^{-1})]. \tag{17}$$

Here we use the relation between the integral of the total mass M and $\langle U \rangle$: $M = 2L\langle U \rangle$.

It follows from Eq. (17) that under assumptions

$$M \leq L \tag{18}$$

and

$$6H_2 \leq M^2L^{-1}, \tag{19}$$

the right-hand side of the inequality (17) is non-negative. Because H_2 and M are the integrals of motion for Eq. (2), it is evident that if the conditions (18) and (19) take place at the initial moment, then they will be satisfied for all t. Thus,

$$R_{tt}R - (1+\alpha)R_t^2 \geq 0, \tag{20}$$

and under additional assumption $R_t > 0$ at the moment $t = 0$, it can be shown [19] that function $A = R^{-\alpha}$ tends to zero as $t \to t_0 = R_0/\alpha R_{0t}$.

It is easy to obtain, in this case, that function R is bounded from below as

$$R(t) \geq R_0 \left(\frac{t_0}{t_0 - t}\right)^{1/\alpha}. \tag{21}$$

Summarizing, we prove that if the initial data satisfy the conditions (a3) $6H_2 \leq M^2L^{-1}$, (b3) $M \leq L$, and (c3) $R_t|_{t=0} > 0$, then the positive function $R(t)$ becomes infinite in a finite time. That corresponds to the appearance of a singularity in the initially smooth solution of Eqs. (1) and (2). The local structure of the solutions in the vicinity of the blow-up point was analyzed in Ref. [15]. For a certain class of solutions of Eq. (2) it may be proved by the IST that near the collapse point the solutions have a self-similar form, locally.

We close this section with a remark that, indeed, a sharper criterion can be achieved by using the inequality $P_2^2 \leq \int_{-L}^{L} U^2 dx \int_{-L}^{L} \Phi^2 dx$ for the estimation in (16) the terms proportional to I_1 and I_2.

IV. BLOW-UP IN THE IMPROVED BOUSSINESQ EQUATION

In this section, we shall present a blow-up theorem for Eq. (1). This case may be handled in the similar way to that for Eq. (2). To the best of our knowledge the analytic proof of the blow-up for Eq. (1) has been unknown even for the case of the infinite interval. It is our aim in this section to present a proof of the blow-up for Eq. (1), both for periodic boundary conditions and for the case of an infinite system.

Let us first treat the periodic case. From the results of the preceding section we are able to presuppose a form of a majoring function for Eq. (1). It is quite reasonable to consider the function g defined by the relation $g_x = U - \langle U \rangle$. Here and throughout this section the function U is a solution of Eq. (1) and $\langle U \rangle = \frac{1}{2L}\int_{-L}^{L} U dx = M/2L$. From the definition, it follows that the function g is of the period $2L$ and it satisfies the equation

$$g_t = Q^{-1}\Psi_x. \tag{22}$$

The main point in the proof is a choice of the majoring function. Let define the quantity $S(t)$ by the relation $S = \int_{-L}^{L} gQg\, dx$, where $Q = -\frac{\partial^2}{\partial x^2} + 1$.

Similarly to the approach used in the preceding section, we assume that at the initial moment $t = 0$ the following conditions are satisfied: (a4) $6H_1 - M^2L^{-1} \leq 0$, (b4) $M \leq L$, and (c4) $S_t|_{t=0} > 0$.

Then, under these assumptions, it is easily demonstrated that $S(t)$ becomes infinite in a finite time.

The proof of this theorem is in every respect similar to the procedure used in the preceding section.

Differentiating S with respect to t we get

$$S_t = 2\int_{-L}^{L} g\Psi_x dx. \qquad (23)$$

Note that the non-negative operator Q may be written in the form $Q = -\frac{\partial^2}{\partial x^2} + 1 = (\frac{\partial}{\partial x} + 1)(-\frac{\partial}{\partial x} + 1) = A^\dagger A$. A similar representation may be found for Q^{-1}. As a result, it can be shown that the right-hand side of Eq. (24) is estimated from above by the following inequality:

$$S_t \le 2S^{1/2}\left(\int_{-L}^{L} \Psi Q^{-1}\Psi dx\right)^{1/2}. \qquad (24)$$

Calculating the second derivative of the function S, we have

$$\begin{aligned} S_{tt} &= 2\int_{-L}^{L}\left(\Psi Q^{-1}\Psi + g\frac{\partial}{\partial x}(U - U^2)\right)dx \\ &= 4J_1 - 2\int_{-L}^{L}(U - \langle U\rangle)(U - U^2)dx. \end{aligned}$$

It follows from the latter equation and inequality (24) that the function S satisfies the majoring inequality:

$$\begin{aligned} S_{tt}S - (1+\alpha)S_t^2 \ge S[&-6H_1 + 10J_1 + 2J_2 + M^2L^{-1} \\ &-2ML^{-1}J_2 - 8(1+\alpha)J_1]. \end{aligned}$$

We still have the freedom to set parameter α to be less then $\frac{1}{4}$. Such a choice leads to the conclusion that under assumptions (a4), (b4), and (c4) function $S(t)$ develops singularity in a finite time. Thus, conditions (a4), (b4), and (c4) determine the initial field distributions which are themselves smooth, but for which the solution that emanates from them becomes infinite in a finite time.

Letting $L \to +\infty$ we obtain similar results for the initial-value problem to Eq. (1) for the infinite interval. For the case of the finite value of the integral $M = \int_{-L}^{L} U dx$ in the limit $L \to +\infty$ it is easy to show that the conditions (a4), (b4), and (c4) result in (a) $H_2 \le 0$ and (b) $S_t|_{t=0} > 0$.

One may observe that on the infinite interval the quantity S makes sense for Eq. (1) only under the additional constraint

$$\int_{-\infty}^{+\infty} U dx = 0. \qquad (25)$$

V. CONCLUSION

We have considered two basic forms of the Boussinesq equation which describe propagation of nonlinear waves in a weakly dispersive medium. We have proved the occurrence of the blow-up for the periodic solutions of the well-posed Boussinesq equation. The sufficient conditions were determined under which the solutions of the initial-value problem with the periodic boundary conditions develop a singularity. For the improved Boussinesq equation we have obtained analytically the sufficient criterion of the blow-up both for the solutions of the problem with the periodic boundary conditions and for the solutions of the initial-value problem on the infinite interval.

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